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CLIMB PATH FOR LEAST ELAPSED TIME (u)

N. C. Peterson

RM-245

7 October 1949

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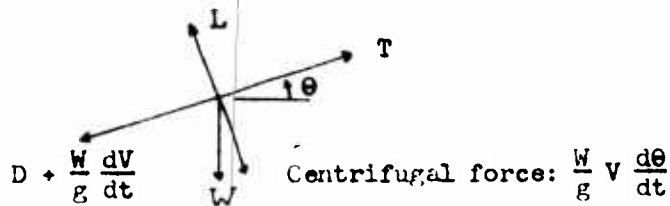
CLIMB PATH FOR LEAST ELAPSED TIME

The purposes of this note are to examine the suitability of an engineering method which has been in use for determining the path a jet-powered airplane should follow to achieve a specified speed and altitude in the least elapsed time, to indicate the conditions under which the method becomes unrealistic, and to suggest an alternative procedure. Ideally, the optimum acceleration-climb path would be obtained as the solution of a variation problem; this problem has been written out in very general form by Magnus R. Hestenes (RAND RM-100). The solution of this problem has not been found. To obtain the method recommended in this paper the variation problem has been expressed in the least general form which still takes account of the important mechanisms influencing the time consumed by the airplane to accelerate and climb. The solution of the simplified problem is then approximated by a systematic construction. In the recommended method the load factor, flight path angle, and the longitudinal acceleration are considered and the elapsed flight time is expressed as the sum of two integrals, one corresponding to level flight acceleration and the other to the climb at (variable) high speed. The optimum path is at hand when the sum of the integrals is a minimum.

While this note was in preparation, the writer had the benefit of several interesting and fruitful discussions of the problem with T. F. Kirkwood.

1. First formulation of the problem.

The present engineering method of determining the optimum acceleration-climb path is now to be considered. The forces on the airplane are depicted on the following diagram.



T = thrust

D = drag

W = weight

L = lift

g = acceleration due to gravity

t = time

V = flight speed (assumed parallel to T)

Apparently

$$T = D + W \sin \theta + \frac{W}{g} \frac{dV}{dt}$$

(1)

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CONFIDENTIAL

- 2 -

hence

$$\frac{(T - D)V}{W} = V \sin \theta + \frac{V}{g} \frac{dV}{dt}$$

Let

$$\begin{aligned} \frac{(T - D)V}{W} &= R_o = \text{rate of climb for zero acceleration} \\ &= \text{"static rate of climb"} \end{aligned}$$

$$V \sin \theta = \frac{dh}{dt} = R = \text{actual rate of climb}$$

and assume that along the flight path

$$V = V(h)$$

Then

$$R_o = R + \frac{V}{g} \frac{dV}{dh} \frac{dh}{dt} = R \left(1 + \frac{V}{g} \frac{dV}{dh} \right)$$

or

$$dt = \frac{\left(dh + \frac{V}{g} dV \right)}{R_o}$$

Thus the time to climb, λ , is

$$\lambda = \int_{h_1, V_1}^{h_2, V_2} \frac{\left(dh + \frac{V}{g} dV \right)}{R_o} = \int_{h_1, V_1}^{h_2, V_2} \frac{\left(1 + \frac{V}{g} \frac{dV}{dh} \right) dh}{R_o} \quad (2)$$

Now $R_o = \frac{(T - D)V}{W}$ where the thrust, T , is a function of altitude, h , and velocity, V , and where the drag, D , depends not only on the altitude and velocity but also on the tangential acceleration and on the lift being furnished by the wing, which in turn is a function of weight, angle of climb, and the load factor (i.e., normal acceleration or centrifugal force). In the method of obtaining the best climb path being scrutinized, it is assumed that R_o is a function of altitude and velocity only, hence the influence of load factor and angle of climb are ignored. It is further assumed that the lift is equal to the weight in the climbing attitude.

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-3-

Consider the numerator on the right in (2):

$$\left(1 + \frac{V}{g} \frac{dV}{dh}\right) dh.$$

On kinematic grounds, it is possible for this expression to vanish, namely whenever the sum of the kinetic and potential energies of the airplane is constant. This occurs physically when the component of the aircraft weight in the flight direction is sustained wholly by inertia. However, when it is assumed that R_o is a function of h and V only, the denominator of (2) does not necessarily vanish with the numerator and according to that expression the airplane can achieve altitude at the expense of speed in zero or negative elapsed time. Obviously, the elapsed time is to some extent underestimated by (2) whenever $\frac{dV}{dh}$ is negative. Conversely, the eventuality $\frac{dV}{dh} > 0$ entails overestimation of of the elapsed time by (2) with $R_o = R_o(h, V)$.

It has been shown that the use of the formula

$$\lambda = \int_{h_1, V_1}^{h_2, V_2} \frac{(dh + \frac{V}{g} dV)}{R_o(h, V)} \quad (2a)$$

for the deduction of optimum flight paths is unrealistic on the following counts: The airplane is restricted to small climb angles, the effects of load factor (centrifugal force) are not accounted for, and the time consumed in gaining altitude at the expense of speed (airplane partially inertia-born) is underestimated. Apparently paths determined by means of (2a) may not be correct.

The engineering procedure for solving (2a) is to construct a chart of R_o versus V with contours of constant h using the characteristics of the specific airplane, and to draw on this chart a set of lines joining the initial and final points of altitude and speed. The elapsed time along each path is calculated by numerical integration of (2a). The trend of the results is a guide to the construction of additional path contours, the process being continued until an optimum path is found. Unfortunately, the paths so obtained are usually such that the conditions under which (2a) is valid and realistic are violated. An improved procedure is needed.

2. Preliminary variation-calculus arguments.

Equation (2) is now exploited for qualitative information. Assume that all operation is at maximum power so that the airplane completes a climb at speeds less

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CONFIDENTIAL

- 4 -

than or equal to the final speed. Also assume initially that h and V are piecewise continuously differentiable functions of a variable ζ ($0 \leq \zeta \leq 1$) such that

$$\left. \begin{aligned} h(0) &= h_1 = \text{initial altitude} \\ h(1) &= h_2 (> h_1) = \text{final altitude} \\ V(0) &= V_1 = \text{initial speed} \\ V(1) &= V_2 = \text{final speed.} \end{aligned} \right\} \quad (4)$$

The problem is then stated in the form: to find functions $h(\zeta)$, $V(\zeta)$ fulfilling (4) and such that

$$\lambda = \int_0^1 \frac{(h' + \frac{V}{g} V') d\zeta}{R_0[h(\zeta), V(\zeta)]} \quad (5)$$

is a minimum.

Upon conduction the first variation of λ with respect to h and V the following simultaneous equations appear

$$\frac{h'}{R_0^2} \left[\frac{\partial R_0}{\partial V} - \frac{V}{g} \frac{\partial R_0}{\partial h} \right] = 0 \quad (6)$$

$$\frac{V'}{R_0} \left[\frac{\partial R_0}{\partial V} - \frac{V}{g} \frac{\partial R_0}{\partial h} \right] = 0 \quad (7)$$

There are apparently two alternatives:

$$a) \quad \frac{\partial R_0}{\partial V} - \frac{V}{g} \frac{\partial R_0}{\partial h} = 0$$

or

$$b) \quad h' = 0 \text{ and } V' = 0 \quad (R_0^2 < \infty)$$

In general (a) is not fulfilled (but is fulfilled in an important consideration to follow), hence option (b) is to be scrutinized. It has been specified that $h_2 > h_1$, therefore in view of (b) there exists no continuous $h(\zeta)$ such that $h(1) > h(0)$ with $h' \equiv 0$ except at a finite number of points; it follows that the problem specified by (1) and (3) has no minimizing solution. However, there does exist a lower limit of the elapsed time, which limit can be approached with arbitrary precision.

A guide to finding the limiting flight path is obtained by relaxing the piecewise continuous differentiability condition on h and V . Then if

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CONFIDENTIAL

- 5 -

$$\frac{\partial R_o}{\partial V} - \frac{V}{g} \frac{\partial R_o}{\partial h} \neq 0 \quad (8)$$

one has that except at a finite number of points

$$h' = 0$$

$$V' = 0$$

$$V = \frac{dV}{dh} \frac{dh}{d\zeta} = \frac{dV}{dh} h' \quad (9)$$

So when $V' = 0$, either $\frac{dV}{dh} = 0$ or $h' = 0$.

$$\text{Also: } h' = \frac{V'}{\frac{dV}{dh}}$$

and when $h' = 0$, either $V' = 0$ or $\frac{dV}{dh} \rightarrow \infty$. (10)

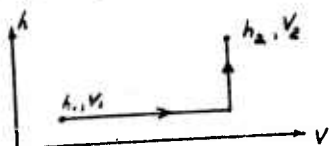
According to (9), a change in altitude occurs at constant ζ , therefore: climb at constant velocity. Similarly by (10), accelerate at constant altitude.

To decide whether to accomplish the changes in speed and altitude in large or small steps, consider again equation (2). The function $R_o(h,V)$ decreases with altitude and, below the speed for maximum static rate of climb, increases with speed. Therefore, to minimize the elapsed time the rules are:

Accelerate at minimum constant altitude

Climb at the maximum constant speed (i.e. V_2)

On the h,V plane, the fundamental flight plan is as sketched:



(V_2 less than speed, at h_2 , for maximum static rate of climb)

There is one curve on the h,V plane on which the following equation holds

$$\frac{\partial R_o}{\partial V} - \frac{V}{g} \frac{\partial R_o}{\partial h} = 0.$$

This curve is usually nearly straight and vertical, and occurs in the region of V max. By the nature of the $R_o(h,V)$ function, the equation can only be fulfilled

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CONFIDENTIAL

- 6 -

when both terms vanish; therefore the curve is the locus of the maximum static rate of climb, for on that locus both $\frac{\partial R_o}{\partial V}$ and $\frac{\partial R_o}{\partial h}$ vanish. Let h_{1c} , V_{1c} be

the point of this locus at h_1 , and h_{2c} , V_{2c} the corresponding point at h_2 ; then if

$$V_1 \leq V_{1c}$$

$$V_2 \geq V_{2c}$$

it is advantageous to accelerate at constant altitude to the point h_{1c} , V_{1c} ; climb along the locus

$$\frac{\partial R_o}{\partial V} = 0, \quad \frac{\partial R_o}{\partial h} = 0$$

to h_{2c} , V_{2c} and then accelerate at constant altitude to V_2 . The case $V_2 < V_1$ is not considered.

It is to be emphasized that the flight paths suggested by the foregoing "rules" are only rough first approximations to the actual optimum paths. These paths are, of course, similar to those obtained by the engineering procedure previously described.

3. Simplest (i.e. least general) realistic formulation of variation problem for acceleration - climb path of least elapsed time.

In this paragraph, a least general realistic formulation of the climb problem is derived. This formulation is apparently too complicated for practical calculation but, in conjunction with the findings in paragraph 2, it motivates a suitable approximation method.

Referring to equation (1)

$$T = D + W \sin \theta + \frac{W}{g} \frac{dV}{dt} \quad (1)$$

Assuming a parabolic airplane polar,

$$D = D_o + \frac{L^2}{\pi q b^2 e} = D_o + D_i \quad (11)$$

where

D_o = parasite drag of airplane

D_i = induced drag

e = airplane efficiency factor

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- 7 -

b = wingspan

q = $\frac{1}{2} \rho V^2$ = dynamic pressure.

The lift is

$$L = W \cos \theta + \frac{W}{g} V \frac{d\theta}{dt}$$

the second term being the centrifugal force. Thus, substituting in (1)

$$T(\rho, V) = D_o(\rho, V) + D_{i_o}(\rho, V) \left(\cos \theta + \frac{V \dot{\theta}}{g} \right)^2 + W \sin \theta + \frac{W}{g} \dot{V} \quad (12)$$

where $D_{i_o} = \frac{W^2}{\pi q b^2 e}$ is the induced drag for level flight at the given speed and

altitude. When the thrust, T, is written as a function of ρ and V only, as in (12), the assumption is tacitly made that the thrust line is essentially parallel to the local tangent to the flight path. It is convenient to make (12) dimensionless by dividing each term by W, and to introduce the notation

$$\begin{aligned} \mathcal{D}_o &= \frac{D_o}{W} = \frac{1}{W} C_{D_o} \frac{1}{2} \rho V^2 S = C_1 \rho V^2 ; C_1 = \frac{C_{D_o} S}{2W} \\ \mathcal{D}_{i_o} &= \frac{D_{i_o}}{W} = \frac{1}{W} \frac{W^2}{\pi b^2 e \frac{1}{2} \rho V^2} = \frac{C_2}{\rho V^2} ; C_2 = \frac{2W}{\pi b^2 e} \\ \tau &= \frac{T(\rho, V)}{W} \end{aligned}$$

Then (12) is

$$\begin{aligned} \tau(\rho, V) &= \mathcal{D}_o(\rho, V) + \mathcal{D}_{i_o}(\rho, V) \left(\cos \theta + \frac{V \dot{\theta}}{g} \right)^2 + \sin \theta + \frac{\dot{V}}{g} \\ &= C_1 \rho V^2 + \frac{C_2}{\rho V^2} \left(\cos \theta + \frac{V \dot{\theta}}{g} \right)^2 + \sin \theta + \frac{\dot{V}}{g} \end{aligned} \quad (13)$$

Equation (13) relates the functions $\rho(t)$, $V(t)$, and $\theta(t)$ along the flight path. The altitude history, $h(t)$, is of course known when either $\rho(t)$ is known (by virtue of the standard atmosphere tables) or $V(t)$ and $\theta(t)$ are known, through the relation

$$h(t) = h_1 + \int_0^t V \sin \theta dt.$$

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- 8 -

The elapsed time increment dt associated with the increments dV , $d\theta$ is found by separating variables in (13) regarded as a differential equation; it is found that

$$gdt = \frac{\frac{2C_2}{\rho V} \cos \theta d\theta + dV + \sqrt{4 \frac{C_2}{\rho V} \cos \theta dV d\theta + dV^2 + 4 \frac{C_2}{\rho} d\theta^2 (\tau - C_1 \rho V^2 - \sin \theta)}}{2 (\tau - C_1 \rho V^2 - \sin \theta - \frac{C_2}{\rho V^2} \cos^2 \theta)}$$

Thus the problem is to determine continuously differentiable functions $\rho(t)$, $V(t)$, $\theta(t)$ such that the elapsed time, λ , is a minimum, where

$$g\lambda = \int_0^{\lambda} \frac{\frac{2C_2}{\rho V} \dot{\theta} \cos \theta + \dot{V} + \sqrt{4 \frac{C_2}{\rho V} \cos \theta \dot{V} \dot{\theta} + (\dot{V})^2 + 4 \frac{C_2}{\rho} \dot{\theta}^2 (\tau - C_1 \rho V^2 - \sin \theta)}}{2 (\tau - C_1 \rho V^2 - \sin \theta - \frac{C_2}{\rho V^2} \cos^2 \theta)} dt. \quad (14)$$

The analysis requires only piecewise continuous differentiability of ρ , V , θ ; but the physical situation deletes the "piecewise". Actually only two of ρ , V , θ are required for a complete solution, the third being available through one of the following equations

$$\rho = \rho(h(t)) ; h = h_1 + \int_0^t V \sin \theta dt$$

$$\theta = \arcsin \frac{\dot{\rho} \frac{dh}{d\rho}(\rho)}{V}$$

$$V = \frac{\dot{\rho} \frac{dh}{d\rho}(\rho)}{\sin \theta}$$

The Euler equations corresponding to (14) are quite complicated. That arising from the variation with respect to ρ is

$$\begin{aligned} & \frac{\dot{V}^2}{\rho} \left[\frac{C_2}{\rho V^2} \cos^2 \theta - C_1 \rho V^2 \right] + \frac{2C_2}{\rho^2 V} \dot{V} \dot{\theta} \cos \theta \left[-C_1 \rho V^2 + \frac{3C_2}{\rho V^2} \cos^2 \theta - \tau + \sin \theta \right] + \\ & + \frac{2C_2 \dot{\theta}^2}{\rho^2} \left[-C_1 \rho V^2 (\tau - 2C_1 \rho V^2 - 2 \sin \theta) + \frac{C_2}{\rho V^2} \cos^2 \theta (3\tau - 2C_1 \rho V^2 - 3 \sin \theta) - \tau^2 + \right. \end{aligned}$$

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- 9 -

$$\sin \theta (2\tau - C_1 \rho V^2 - \sin \theta) + \left[\frac{2C_2 \dot{\theta} \cos \theta}{\rho} \left(2C_1 V + \frac{\sin \theta - \tau}{\rho V} \right) + \dot{V} \left(C_1 V^2 - \frac{C_2}{\rho V^2} \cos^2 \theta \right) \right]$$

$$\sqrt{4\dot{V} \frac{C_2}{\rho V} \dot{\theta} \cos \theta + \dot{V}^2 + 4 \dot{\theta}^2 \frac{C_2}{\rho} (\tau - C_1 \rho V^2 - \sin \theta)} = 0 \quad (15)$$

The equations corresponding to the variations of V and θ are of comparable complexity. Inasmuch as the constants C_1 , C_2 , and the dimensionless thrust function, τ , are not universal but are characteristics of each particular airplane, it is obvious the general integration of the three simultaneous equations of type (15) is impossible, and that their specific integration as a routine calculation is out of the question. This observation motivates the derivation of a new numerical experimental method of the type described in paragraph (1), which however takes into account the important influences of climb angle and load factor. And one has reason to feel that such methods constitute the only practical approach to the problem.

4. Derivation of recommended procedure.

According to paragraph (2), the optimum path is expected to consist of a level portion at low altitude during which the airplane accelerates to a speed near the desired final speed, followed by a climbing portion at rather high speed (not necessarily constant) and terminated by a pull-up if the desired final speed is less than the climbing speed, or by a levelling-out if the final speed is to be greater than the climbing speed. The elapsed times during the level and climbing phases are now separately expressed, and the optimum path is at hand when the sum of the expressions is a minimum.

On the low-and-level phase, it is assumed that $\theta = \dot{\theta} = 0$, hence, according to equation (13) the time consumed in level acceleration from the initial velocity V_1 to the unknown initial climbing velocity V_{1c} is

$$\lambda_a = \frac{1}{g} \int_{V_1}^{V_{1c}} \frac{dV}{\tau(\rho_1, V) - \sigma_o(\rho_1, V) - \delta_{10}(\rho_1, V)} \quad (16)$$

Now the rate of climb is $V \sin \theta$, hence referring again to equation (13), the elapsed time in the climb phase is

$$\lambda_c = \int_{h_1}^{h_2} \frac{dh}{V \left[\tau(\rho, V) - \sigma_o(\rho, V) - \delta_{10}(\rho, V) \left(\cos \theta + \frac{V \dot{\theta}}{g} \right)^2 - \frac{V}{g} \right]} \quad (17)$$

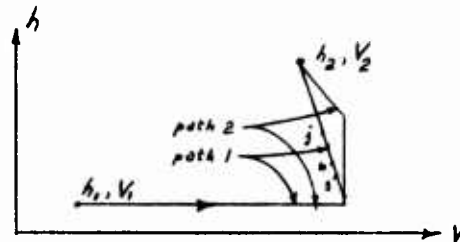
$$\text{and } \lambda = \lambda_a + \lambda_c.$$

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- 10 -

The procedure is as follows: On a plane whose coordinates are altitude and speed, the initial and final points of the acceleration-climb path are plotted and are joined by path lines of the kind shown in the diagram below.



The approximate elapsed time formula is then integrated numerically along the legs of the paths, and the trend of the sum $\lambda_a + \lambda_c$ is a guide to the construction of improved path lines. The process is continued until an apparent minimum is discovered. The optimum paths for different airplanes having the same type of propulsion system have of course a generic similarity which, when known, permits the construction of a first-guess path that is near the optimum and thereby minimizes the computation labor required for the solution.

Numerical integration by either the trapezoidal or Simpson rules is envisaged. The calculation of the integrand of (16) at selected points on the level path line is straightforward, but in the consideration of the climbing phase, several difficulties arise:

a. At the beginning of the climb phase the denominator of (17), which is equal to $V \sin \theta$, is zero because θ is zero throughout the level phase. When θ is zero, so also is dh ; this indeterminacy must be removed in order to start the numerical integration. It is satisfactory to assume that during the first moments of the climb the airplane speed is constant, $\sin \theta = \theta$, $\cos \theta = 1 - \frac{\theta^2}{2}$, and

that the rate of climb (and hence θ) is quadratic in elapsed time. Thus, measuring time from the beginning of the climb

$$V \sin \theta = \beta t^2$$

$$\Delta h = \int_0^t V \sin \theta \, dt = \frac{\beta t^3}{3}$$

At the time t

$$\frac{V \sin \theta}{\Delta h} = \frac{\beta t^2}{\frac{1}{3}\beta t^3} = \frac{3}{t}$$

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-11-

hence

$$t = \frac{3\Delta h}{V \sin \theta}$$

Also,

$$\theta = \alpha t^2 = \frac{t}{2} \dot{\theta}$$

so

$$\theta = \frac{3}{2} \dot{\theta} \frac{\Delta h}{V \theta}$$

which implies

$$\dot{\theta} = \frac{2V\theta^2}{3\Delta h}$$

where all quantities in the last equation are measured at the time t .

Under the stated assumptions, equation (13) is

$$\begin{aligned} \tau &= \delta_o + \delta_{io} \left(1 - \frac{\theta^2}{2} + \frac{V\theta}{g} \right)^2 + \theta \\ &= \delta_o + \delta_{io} \left[1 + \left(\frac{4V^2}{3g\Delta h} - 1 \right) \theta^2 \right] + \theta \end{aligned}$$

Thus at time t

$$\theta = \frac{-1 + \sqrt{1 + \left(\frac{16V^2}{3g\Delta h} - 4 \right) \delta_{io} (\tau - \delta_o - \delta_{io})}}{\left(\frac{8V^2}{3g\Delta h} - 2 \right)} \quad (18)$$

$$\dot{\theta} = \frac{2V\theta^2}{3\Delta h} \quad (19)$$

and the time consumed by the maneuver is

$$t = \frac{2\theta}{\dot{\theta}} = \frac{3\Delta h}{V\theta} \quad (20)$$

It is convenient to select a Δh of about 500 feet; all other quantities in these expressions are then known in terms of the known V and h . One is now in a position to continue the integration of (17) along the selected climb path.

b. At subsequent points (labelled 5, 6, 7, ..., j , ... on the accompanying figure) along the path line on the h, V plane, only h, V , and $\frac{dh}{dV}$ are definitely known, but $\theta, \dot{\theta}$, and $\ddot{\theta}$ are required as well in order to calculate the integrand in (17) at these points. Following a suggestion by Kirkwood, these quantities may be estimated by a continuing integration of θ as follows:

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- 12 -

At point j : h , V , and $\frac{dh}{dV}$ are known, and at point $j - 1$, all required quantities are known.

Let

$$\left. \frac{dh}{dV} \right|_j = \frac{1}{\eta} \quad (\text{a negative quantity in the figure as drawn.})$$

Then at j

$$\dot{h} \equiv V \sin \theta = \frac{1}{\eta} \dot{V} \implies \dot{V} = \eta V \sin \theta \quad (21)$$

Now

$$\tau = \sigma_0 + \sigma_{10} \left(\cos \theta + \frac{V \dot{\theta}}{g} \right)^2 + \sin \theta + \frac{\dot{V}}{g} \quad (13)$$

so at j

$$\tau = \sigma_0 + \sigma_{10} \left(\cos \theta + \frac{V \dot{\theta}}{g} \right)^2 + \sin \theta \left(1 + \frac{\eta V}{g} \right)$$

Thus

$$\dot{\theta}_j = \frac{g}{V} \left(\sqrt{\frac{\tau - \sigma_0 - \sin \theta_j \left(1 + \frac{\eta V}{g} \right)}{\sigma_{10}}} - \cos \theta_j \right) \quad (22)$$

Provided that adequately small intervals of altitude (say 1 or 2 thousand feet) separate successive points $j - 1$, j , ..., a linear variation of θ in each interval may be assumed. Then

$$\theta_j = \theta_{j-1} + \frac{\dot{\theta}_j + \dot{\theta}_{j-1}}{2} \cdot \frac{\Delta h}{\frac{(V \sin \theta)_{j-1} + (V \sin \theta)_j}{2}}$$

hence

$$\dot{\theta}_j = \frac{(\theta_j - \theta_{j-1}) [(V \sin \theta)_j + (V \sin \theta)_{j-1}]}{\Delta h} - \dot{\theta}_{j-1} \quad (23)$$

The equation obtained by comparing expressions (22) and (23) may be solved for θ_j by cut-and-try, whereupon either expression may be used to calculate $\dot{\theta}_j$. Finally, by (21)

$$\dot{V}_j = \eta (V \sin \theta)_j$$

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- 13 -

When climbing at constant velocity, η is replaced by zero in the foregoing equations.

All the tools needed for the systematic comparison of selected acceleration-climb paths are now at hand. A set of calculations using these methods is in preparation and will constitute a subsequent report.

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